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BOUNDS ON THE NUMBER OF CONJUGACY CLASSES OF THE SYMMETRIC AND ALTERNATING GROUPS

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ABSTRACT. Let G be a finite group with Sylow subgroups P_1, \dots, P_n , and let $k(G)$ denote the number of conjugacy classes of G . Pyber asked if $k(G) \leq \prod_{i=1}^n k(P_i)$ for all finite groups G . With the help of GAP, we prove that Pyber's inequality holds for all symmetric and alternating groups.

1. INTRODUCTION

L. Pyber submitted Problem 14.71 to the Kourovka Notebook [4], which reads: “Let $k(H)$ denote the number of conjugacy classes of a group H , and G be a group with Sylow p -groups P_1, \dots, P_n . Prove or disprove: $k(G) \leq k(P_1) \cdot \dots \cdot k(P_n)$.” We will let $k_p(G)$ denote $k(P)$ for a Sylow p -subgroup P of G , which means that Pyber's inequality can be restated as $k(G) \leq \prod_{p| |G|} k_p(G)$. We will verify that this inequality holds if G is a symmetric group S_n or alternating group A_n . Our strategy is to use estimates from analytic number theory to show that that Pyber's inequality holds for $n \geq 60,000$ and a GAP [2] calculation to show the result holds for $n < 60,000$.

2. MAIN RESULTS

Let $p(n)$ denote the number of partitions of an integer n . It is well-known that $k(S_n) = p(n)$ and $k(A_n) \leq 2k(S_n) = 2p(n)$ (see [5, Section 11.1], for instance). Additionally, every Sylow p -subgroup P of S_n is a Sylow p -subgroup of A_n if p is odd, and every Sylow 2-subgroup of S_n has order twice that of every Sylow 2-subgroup of A_n .

Now let $[n]_p$ denote the largest power of p that divides n for a positive integer n and prime p . The first proposition is a bound due to Hall, and the second proposition is the result of a simple GAP script.

Proposition 2.1. [3, Chapter V.15.2] *Let P be a finite group of order p^{2m+e} for some prime p , nonnegative integer m , and $e \in \{0, 1\}$. Then $k(P) \geq p^e + (p^2 - 1)m$.*

Proposition 2.2. *If $n < 60,000$, then $2p(n) \leq \prod_{p| \frac{n!}{2}} (p^{e_p} + (p^2 - 1)m_p)$, where $[\frac{n!}{2}]_p = p^{2m_p + e_p}$ for some m_p and e_p with $e_p \in \{0, 1\}$ for all primes p dividing $\frac{n!}{2}$.*

Proposition 2.3. *If $n < 60,000$, then $k(S_n) \leq \prod_{p| n!} k_p(S_n)$ and $k(A_n) \leq \prod_{p| \frac{n!}{2}} k_p(A_n)$.*

Proof. The result is easy to check if $n \in \{1, 2, 3\}$, so assume that $4 \leq n < 60,000$. The right-side of the inequality from Proposition 2.2 exactly bounds $\prod_{p| |A_n|} k_p(A_n)$ by Proposition 2.1, and so we have $k(A_n) \leq 2p(n) \leq \prod_{p| |A_n|} k_p(A_n)$ for all $n < 60,000$.

Because the bounds in Proposition 2.1 are only a function of the order of the Sylow subgroup, the lower bound from Proposition 2.1 for $k_2(A_n)$ is also a lower bound for $k_2(S_n)$. Then we use the notation and result from Proposition 2.2 and the fact that Sylow p -subgroups of A_n have the same order as Sylow p -subgroups of S_n for odd p to get

$$\begin{aligned}
k(S_n) &< 2p(n) \\
&\leq \prod_{p \mid (n!/2)} (p^{e_p} + (p^2 - 1)m_p) \\
&\leq (2^{e_2} + (2^2 - 1)m_2) \prod_{\substack{p \mid (n!/2) \\ p \text{ odd prime}}} (p^{e_p} + (p^2 - 1)m_p) \\
&\leq k_2(S_n) \prod_{\substack{p \mid n! \\ p \text{ odd prime}}} (p^{e_p} + (p^2 - 1)m_p) \\
&\leq k_2(S_n) \prod_{\substack{p \mid n! \\ p \text{ odd prime}}} k_p(S_n) \\
&\leq \prod_{p \mid n!} k_p(S_n).
\end{aligned}$$

□

It remains to show that the result holds for $n \geq 60,000$.

Proposition 2.4. *If $n \geq 60,000$, then $k(S_n) \leq \prod_{p \mid n!} k_p(S_n)$ and $k(A_n) \leq \prod_{p \mid \frac{n!}{2}} k_p(A_n)$.*

Proof. The center of a nontrivial p -group is nontrivial, so we have $k_p(G) \geq p$ for all primes p dividing $|G|$. Thus, we have

$$\prod_{p \mid n!} k_p(S_n) \geq \prod_{p \mid n!} p \geq 2^{\pi(n)},$$

where $\pi(n)$ is the number of primes that are at most n . Similarly, $\prod_{p \mid \frac{n!}{2}} k_p(A_n) \geq 2^{\pi(n)}$. Further, $\pi(n) \geq \frac{n}{6 \log n}$ by [1, Theorem 4.6], so $\prod_{p \mid n!} k_p(S_n)$ and $\prod_{p \mid \frac{n!}{2}} k_p(A_n)$ are both bounded below by $2^{\frac{n}{6 \log n}}$.

As previously stated, both $k(S_n)$ and $k(A_n)$ are bounded above by $2p(n)$. By [1, Theorem 14.5], $p(n) < e^{\pi \sqrt{\frac{2n}{3}}}$, so we have both $k(S_n)$ and $k(A_n)$ are at most $2e^{\pi \sqrt{\frac{2n}{3}}}$. It remains to show that $2e^{\pi \sqrt{\frac{2n}{3}}}$ is at most $2^{\frac{n}{6 \log n}}$. We let

$$f(n) = \left(2^{\frac{n}{6 \log n}}\right) \left(e^{\pi \sqrt{\frac{2n}{3}}}\right)^{-1}.$$

It is easy to check that $f(60,000) \approx 5.45 \geq 2$ and f is increasing for $n \geq 60,000$, so $f(n) \geq 2$ and

$$2e^{\pi \sqrt{\frac{2n}{3}}} \leq 2^{\frac{n}{6 \log n}}$$

for $n \geq 60,000$. We conclude that if G is S_n or A_n , then

$$k(G) \leq 2p(n) \leq 2e^{\pi \sqrt{\frac{2n}{3}}} \leq 2^{\frac{n}{6 \log n}} \leq 2^{\pi(n)} \leq \prod_{p \mid |G|} k_p(G).$$

□

Propositions 2.3 and 2.4 imply our final theorem.

Theorem 2.5. *If n is any positive integer, then $k(S_n) \leq \prod_{p \mid n!} k_p(S_n)$ and $k(A_n) \leq \prod_{p \mid \frac{n!}{2}} k_p(A_n)$.*

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